Universality in quantum parametric correlations

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We investigate the universality of correlation functions of chaotic and disordered quantum systems as an external parameter is varied. A general scaling procedure is introduced which makes the theory invariant under reparametrizations. Under certain general conditions we show that this procedure is unique. The approach is illustrated with the particular case of the distribution of eigenvalue curvatures. We also derive a semiclassical formula for the nonuniversal scaling factor, and give an explicit expression valid for arbitrary deformations of a billiard system. [S1063-651X(99)10910-3]

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Several criticisms were formulated many years ago concerning the applicability of random matrix theory (RMT) in the description of the behavior of complex quantum systems such as the atomic nucleus. The basic question was as follows: should one trust the predictions of RMT at all if already the average density of states (Wigner's semicircle law) does not give a good description of real systems? We now know that RMT describes the universal behavior of local fluctuation properties. These have been shown to be, in the limit of large dimensions, invariant for a large class of ensembles of matrices, while the average spectral density is ensemble dependent and therefore nonuniversal [1,2]. For a given physical system, it is now well established that in order to eliminate the system-dependent features and observe universal fluctuations one should consider, instead of the original eigenenergies E_j, $j=1,2,\ldots$, the unfolded spectrum

$$\epsilon_j = \bar{N}(E_j), \qquad (1)$$

where

$$\bar{N}(E) = \int^E \bar{\rho}(E')dE' \qquad (2)$$

is the integrated average density of states [3]. By construction the new energies ϵ_j have unit mean level spacing. If the system has a classical analog with n degrees of freedom described by the Hamiltonian $H(\vec{p},\vec{q})$, $\vec{p}=(p_1,\ldots,p_n)$, $\vec{q}=(q_1,\ldots,q_n)$, then to leading order in a semiclassical expansion the average density of states is given by the Thomas-Fermi approximation

$$\bar{\rho}(E) \approx \frac{1}{(2\pi\hbar)^n} \int\int d^np\, d^nq\, \delta(E-H(\vec{p},\vec{q})). \qquad (3)$$

This expression allows for an explicit implementation of the unfolding procedure (1)–(2).

More recently, extensions of the universal behavior of disordered as well as chaotic quantum systems have been developed to include parametric correlations and fluctuations of the energy levels as some real parameter X controlling the dynamics is varied (see Sec. III-H of [4] for a recent review). These extensions are of physical importance because many response functions such as the magnetic susceptibility or the conductance may be expressed as parametric correlations. Exactly as for the usual fluctuations computed at fixed parameter values, the mean properties of the flow of energy levels $\epsilon_j(X)$ when X is varied are system dependent and therefore nonuniversal. An appropriate scaling procedure is again necessary in order to extract the universal behavior.

While the average velocity of the levels is set to zero if the energy unfolding (1) is implemented for arbitrary X, what now characterizes the flow $\epsilon_j(X)$ is the typical slope of the eigenvalues with respect to X. This typical slope is not only system dependent, but it is moreover parameter dependent. In fact, the curves $\epsilon_j(X)$ as well as their average properties vary when expressed as a function of a different parameter $Y(X)$. Therefore, in order to have universality in the theory it is necessary to show the existence of a parametric representation of the elementary quantum objects satisfying two basic conditions: (i) it scales out system-dependent features and (ii) it is invariant under reparametrizations. This particular parametric representation should moreover be unique (it should be ''the'' parametrization), since the existence of several different invariant parameters associated with X would destroy the universality of the correlators.

The way to construct such a parameter is as follows. Consider the variance of the distribution of the parametric velocities computed at fixed X in some window ($\epsilon-\Delta\epsilon/2$, $\epsilon+\Delta\epsilon/2$), containing N levels, in which the statistics are evaluated,

$$\langle v_X^2 \rangle = (1/N)\sum_j (\partial\epsilon_j/\partial X)^2. \qquad (4)$$

This function characterizes the nonuniversal mean properties of the flow $\epsilon_j(X)$ in the window we are considering. More precisely, it characterizes the response of the energy levels to an external perturbation and, following Thouless [5], may be interpreted as a generalized conductance of the system [6,7].

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Then, to scale the spectral flow and eliminate the system-dependent characteristics we introduce the parameter

$$\mu = \int_{X_0}^{X} \sqrt{\langle v_X^2 \rangle} \, dX, \tag{5}$$

where X_0 is some reference value. The definition of μ is universal, independent of the nature of X. If the system is reparametrized by introducing a new parameter $Y(X)$, then $\sqrt{\langle v_X^2 \rangle} = \sqrt{\langle v_Y^2 \rangle} |\partial Y / \partial X|$. The last factor $|\partial Y / \partial X|$ is compensated by the Jacobian of the transformation in Eq. (5) and we have

$$\mu = \int_{X_0}^{X} \sqrt{\langle v_X^2 \rangle} \, dX = \int_{Y_0}^{Y} \sqrt{\langle v_Y^2 \rangle} \, dY \tag{6}$$

for any transformation $Y(X)$. Notice that the prescription (5) yields a parametrization that makes the velocity variance identical to one, $\langle v_\mu^2 \rangle \equiv 1$, at all parameter values μ.

We investigate now the uniqueness of this parametrization. It is actually possible to construct an infinite number of invariant parameters, according to

$$\mu_m = \mathcal{N}_m \int_{X_0}^{X} \langle v_X^{2m} \rangle^{1/2m} \, dX,$$

with $m = 1, 2, \ldots$ [all these parameters satisfy the analog of Eq. (6)]. The normalization constant $\mathcal{N}_m$ is defined in terms of the $2m$th moment of a Gaussian distribution with variance one, $\mathcal{N}_m = [(2m-1)!!]^{-1/2m}$. In general these parameters define different functions of X. However, there is one particular case for which they are all identical [and coincide with the simplest one $m=1$ of Eq. (5)], and this is when the distribution of velocities is Gaussian. This distribution is expected to hold for generic fully chaotic systems. It can be easily seen to apply to the following parametric random matrix model:

$$H = \cos X H_1 + \sin X H_2, \tag{7}$$

where H_1 and H_2 are two independent random matrices belonging to one of the three universality classes $\beta = 1, 2$, or 4 (orthogonal, unitary, and symplectic, respectively) [8]. Furthermore, a Gaussian distribution of velocities also holds for weakly disordered metallic systems, where it has been explicitly demonstrated [7].

We have therefore established the existence of a unique, *parametric-invariant scaling* procedure for the restricted class of systems having a Gaussian distribution of velocities. Conversely, this distribution of velocities becomes a necessary condition for universality. In nongeneric fully chaotic or disordered systems where the velocities are not Gaussian distributed (like, for example, in strongly disordered electronic systems or banded random matrix models [9]) the different invariant parametrizations are not equivalent, and the universality is lost.

All these considerations have their analog in the usual unfolding procedure at fixed parameter values. The motivation to unfold the spectrum according to the prescription (1)–(2)—and not according to the more ''primitive'' one $\bar{\rho}(E_j)E_j$—is not only because it fixes the average mean spac-

ing to one (both prescriptions do), but more basically because it makes the new spectrum ϵ_j invariant under reparametrizations of the energy.

As an illustration let us consider two well-known correlators, the velocity correlation function

$$c_\mu(\nu) = \left\langle \frac{\partial \epsilon_j}{\partial \mu}(\mu_0) \frac{\partial \epsilon_j}{\partial \mu}(\mu_0 + \nu) \right\rangle_{\mu_0, j}, \tag{8}$$

and the distribution $p(k_\mu)$ of curvatures

$$k_\mu = \frac{1}{\pi \beta} \frac{\partial^2 \epsilon_j}{\partial \mu^2}. \tag{9}$$

These quantities were investigated in the past for fully chaotic systems and in random matrix theory, as well as in disordered systems [10,6,11,7,12–16]. The parametric correlators were computed in terms of the rescaled parameter

$$x = \sqrt{\langle v_X^2 \rangle} X, \tag{10}$$

which, in general, is not invariant under reparametrizations and may produce nonuniversal results. For example, curvatures with respect to the parameter μ and x [$k_x = (\partial^2 \epsilon_j / \partial X^2)/(\pi \beta \langle v_X^2 \rangle)$] are related by

$$k_\mu = k_x - \frac{(\partial \epsilon_j / \partial X)}{2\pi \beta} \frac{(\partial \langle v_X^2 \rangle / \partial X)}{\langle v_X^2 \rangle^2}. \tag{11}$$

Thus, what is expected to be universal is not the distribution of k_x but the distribution of the particular combination given on the right-hand side (rhs) of Eq. (11). The lack of reparametrization invariance of k_x was properly emphasized and nicely illustrated in Ref. [15].

It follows from the definition (10) that the parameter x coincides with μ if the function $\langle v_X^2 \rangle$ is stationary with respect to X, i.e., it is independent of X. Computations done on a stationary spectrum having a Gaussian distribution of velocities using the parameter x are therefore correct in the sense that the results obtained are expected to be universal. Because the Hamiltonian (7) satisfies this property, the distribution

$$p(k) = \mathcal{N}_\beta (1 + k^2)^{-(\beta + 2)/2} \tag{12}$$

obtained from that model in Ref. [14] *is* the universal distribution for the curvature (here $\mathcal{N}_\beta$ is a normalization constant). On the other hand, in the generic situation when parametric correlations are computed in a system where $\langle v_X^2 \rangle$ varies with X (nonstationary spectrum), the use of the rescaling (10) leads to nonuniversal results and the observed distribution changes with X exactly as observed in Ref. [15], unless the correct parametric-invariant scaling (5) is used. To illustrate this point we show in Fig. 1 the curvature distribution of k_μ and compare it with that of k_x for the Robnik-limaçon billiard (the ''worst case'' found in Ref. [15] is considered). The use of the universal parameter μ produces a dramatic change on the distribution. We believe that the agreement with Eq. (12) will be further improved by going higher in the spectrum.
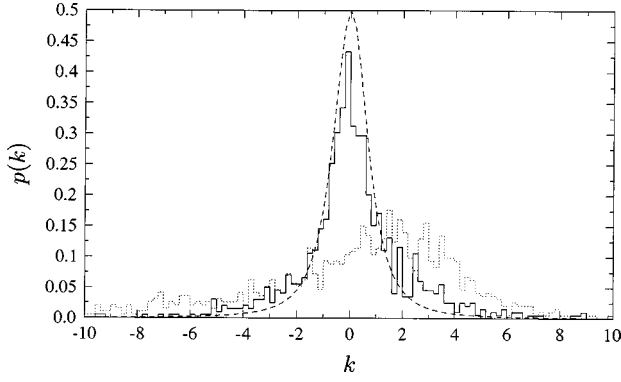
FIG. 1. Curvature distributions from the eigenvalues $n = 2001$ to $n = 3000$ of the odd symmetry class of the Robnik-limaçon billiard at $X = \lambda = 0.49$; dotted histogram, k_x (noninvariant); continuous histogram, k_μ (invariant). Dashed curve, RMT prediction, Eq. (12).

The functional values of statistics involving only first derivatives with respect to the parameter, like in Eq. (8), are invariant under reparametrizations using the parameter x, and one would then believe that both scalings are equivalent in this particular case. However, if $\langle v_X^2 \rangle$ varies with X the function looks different when plotted against x or μ, because $x(X)$ is different from $\mu(X)$. The use of the appropriate parametric-invariant scaling (5) is therefore necessary even for correlators involving first derivatives only.

As for the density of states Eq. (3), it would be desirable to have an explicit expression allowing for a direct computation of the nonuniversal function $\langle v_X^2 \rangle$ (the generalized conductance). Such an expression may be obtained from semiclassical estimates of off-diagonal matrix elements by applying results of RMT [8] or by comparing semiclassical computations of the parametric density correlation function with results obtained in disordered metallic systems [7,17]. It is, however, possible to derive it from a direct semiclassical calculation based on the Gutzwiller trace formula. The starting point is the counting function for the unfolded spectrum $N(\epsilon, X) = \sum_j \Theta(\epsilon - \epsilon_j(X))$ (Θ is the Heaviside function). We define the velocity density as

$$\rho_v^\eta(\epsilon) = -\frac{dN^\eta}{dX} = \frac{1}{\pi} \sum_j \frac{\eta}{(\epsilon - \epsilon_j)^2 + \eta^2} \left(\frac{\partial \epsilon_j}{\partial X} \right), \quad (13)$$

where for convenience we have replaced the delta function by an η-smoothed Lorentzian. From this we obtain

$$\sum_j \delta(\epsilon - \epsilon_j)(\partial \epsilon_j/\partial X)^2 = \lim_{\eta \to 0} 2\pi \eta [\rho_v^\eta(\epsilon)]^2, \quad (14)$$

which is our starting point for the semiclassical calculations since the average of the lhs defines the average variance $\langle v_X^2 \rangle$. To leading order in $\hbar$, $N^\eta(\epsilon, X)$ for a chaotic system is given by [18]

$$N^\eta(\epsilon, X) = \epsilon + \frac{\hbar}{i} \sum_p \frac{A_p}{T_p} e^{iS_p/\hbar - \eta t_p/\hbar}.$$

The sum is over all the periodic orbits of the classical system, A_p is an amplitude that depends on their stability, and S_p and T_p are their action and period, respectively. Further-

more, $t_p = |\partial S_p/\partial \epsilon| = |T_p/\bar{\rho}|$ and the sum runs over positive and negative values of p. Deriving this expression with respect to X according to Eq. (13), replacing in Eq. (14), averaging in a small energy window, and keeping only the diagonal part of the sum one obtains a semiclassical estimate for $\langle v_X^2 \rangle$. An analogous calculation was made in Ref. [19] to compute the variance of diagonal matrix elements of an arbitrary operator A. In our case $A = \partial H/\partial X$, and we obtain

$$\langle [\rho_v^\eta(\epsilon)]^2 \rangle = \frac{2}{h^2} \int_0^\infty \frac{K_D}{T^2} \langle Q_p^2 \rangle e^{-2\eta T/(\bar{\rho}\hbar)} dT, \quad (15)$$

with $K_D(T) = h^2 \langle \sum_p |A_p|^2 \delta(T - T_p) \rangle$, and $\langle Q_p^2 \rangle = \sum_p Q_p^2 |A_p|^2 / \sum_p |A_p|^2$ with $Q_p = \partial S_p/\partial X|_\epsilon$. The sum runs over orbits having a period T_p between T and $T + dT$. It can be shown that the vanishing of the off-diagonal terms follows from RMT, if we assume that the semiclassical theory can reproduce the RMT results for parametric correlations. Furthermore, the variance of the distribution of Q_p is simply proportional to the period of the orbit [20], $\langle Q_p^2 \rangle = \alpha T$.

The function $K_D(T)$ has also a linear dependence on the period [21], $K_D(T) = 2T/\beta$. From this and from Eqs. (15) and (14) we get the final expression for the variance of the velocity [19],

$$\langle v_X^2 \rangle = \frac{\alpha \bar{\rho}}{\beta \pi \hbar}. \quad (16)$$

It is, moreover, easy to see from the previous expressions that with our assumptions there is no contribution to Eq. (16) coming from the nonuniversal short periodic orbits.

The classical parameter α in Eq. (16) is generally, like $\bar{\rho}$, a function of X and of the energy. It depends on the system and the particular parametric variation under study. For example, for a two-dimensional billiard of area $\mathcal{A}$ and perimeter $\mathcal{L}$, consider a general deformation that moves the boundary (parametrized by s, $0 \leq s < \mathcal{L}$) by an amount $g(s)dX$ in the normal direction. The quantities Q_p for periodic orbits with period T are given in this case by

$$Q_p = -\frac{T\mathcal{L}\hbar^2 k^2}{2m\mathcal{A}} \langle g \rangle + 2\hbar k \sum_{i=1}^{n_p} \cos \theta_i \, g(s_i), \quad (17)$$

where $\langle f \rangle = (1/\mathcal{L}) \int_0^\mathcal{L} f(s) ds$, k is the wave number corresponding to ϵ, n_p is the number of bounces of the orbit, and s_i and θ_i denote the points of reflection and the angle of the trajectory with the tangent to the boundary at these points, respectively. After calculating $\langle Q_p^2 \rangle$ from this expression we obtain the following result for α:

$$\alpha = \frac{4\mathcal{L}(2mE)^{3/2}}{\pi m \mathcal{A}} (\mathcal{C}_1 \langle g^2 \rangle - \mathcal{C}_2 \langle g \rangle^2). \quad (18)$$

The constants $\mathcal{C}_1$ and $\mathcal{C}_2$ contain dynamical information about the periodic orbits,

$$\mathcal{C}_1 = \frac{2}{3} \left[1 + 2 \sum_{\tau=1}^\infty f(\tau) \right], \quad (19)$$

$$\mathcal{C}_2 = \frac{\pi^2}{16}\left[1 + 2\sum_{\tau=1}^{\infty} f(\tau) - \frac{\sigma_{n_p}^2}{\overline{n_p}}\right], \qquad (20)$$

with

$$f(\tau) = \frac{\langle g(s_i)\cos\theta_i g(s_{i+\tau})\cos\theta_{i+\tau}\rangle - (\pi^2/16)\langle g\rangle^2}{(2/3)\langle g^2\rangle - (\pi^2/16)\langle g\rangle^2}. \qquad (21)$$

The numerical factors in these expressions come from the ergodic phase-space averages $\langle\cos\theta\rangle^2 = \pi^2/16$ and $\langle\cos^2\theta\rangle = 2/3$. By definition, $f(0) = 1$ and $f(\tau)\to 0$ for large τ. The quantities $\sigma_{n_p}^2$ and $\overline{n_p}$ are the variance and the average of the number of bounces of periodic orbits with period $T\to\infty$, respectively. In the extreme case where the correlations between the consecutive segments of an orbit are neglected (the uncorrelated, ''random'' case), then $\mathcal{C}_1 = 2/3$ and $\mathcal{C}_2 = 0$.

Notice from Eqs. (16) and (18) the energy dependence $E^{3/2}$ of the average variance of the velocity [16] (to leading order $\bar{\rho} = \mathcal{A}/4\pi$ for two-dimensional billiards in dimensionless units $\hbar = 2m = 1$). This dependence holds for arbitrary deformations of chaotic two-dimensional billiards, but other dynamical properties of a system or other parametric variations may produce different energy dependences (for example, for integrable billiard systems we obtain $\langle v_X^2\rangle \propto E^2$, while for chaotic billiards with Aharonov-Bohm fluxes $\langle v_X^2\rangle \propto \sqrt{E}$ [16]). Figure 2 shows the (normalized) variance of the velocity for a chaotic billiard in comparison with the semiclassical approximation (18) with $\mathcal{C}_1 = 2/3$ and $\mathcal{C}_2 = 0$. A linear increase is observed, in agreement with the predicted energy dependence. Moreover, the slope agrees with the simple semiclassical estimate. The variation with energy also implies that in general it is not possible to simultaneously set to one the mean spacing and $\langle v_X^2\rangle$ for arbitrary energies. All parametric correlators should therefore be computed in an energy window.

In conclusion, we have introduced a universal invariant way of scaling the parameter-dependent correlators in quan-
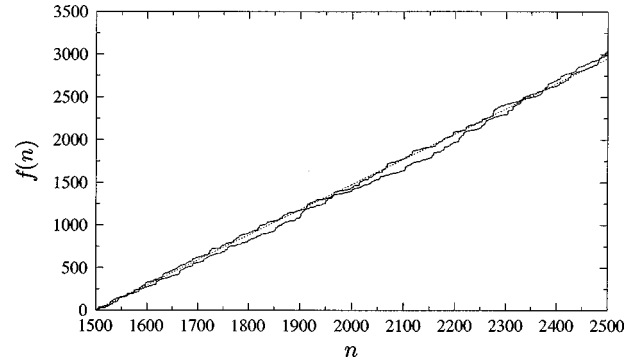

FIG. 2. The quantity $f(n) = \sum_{j=n_0}^{n}(\partial\epsilon_j/\partial r)^2/\epsilon_j^{3/2}$, in dimensionless units, for the eigenvalues of the two symmetry classes of a quarter Sinai billiard at radius $r = 0.5$ and $n_0 = 1500$ (full lines) in comparison with the semiclassical approximation (dotted line).

tum mechanics. We have, moreover, obtained expressions that allow for an explicit implementation of this scaling procedure and predicted a universal energy dependence of the variance of the velocities for deformations of a chaotic billiard. Contrary to the density of states, which depends only on global geometrical properties of phase space, the derivation of Eq. (16) assumes a (fully) chaotic dynamics. Moreover the parametric-invariant scaling procedure is unique only in the situation of generic chaotic and weakly disordered systems with a Gaussian distribution of the velocities. Our results, in particular, the predictions concerning two-dimensional billiards, may be tested experimentally in quantum dots, metallic grains, or microwave cavities.

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